Linear Difference Equation of Higher Order (Continued)

Recall : Casoration
Fundamental set of eqn
General solution

Example

Consider the difference equation
$$x(n+3) - 7x(n+1) + 6x(n) = 0$$
Show that the sequences $1, (-3)^n$ and 2^n are solutions of the equation.

Also find the casoration of the given solution. Hence show that the solutions form a fundamental set.

Ans: $x(n)=1$ is a solution, since

$$1-7+6=0$$

Also, $x(n)=(-3)^n$ is a solution,

since

$$(-3)^{n+3}-7(-3)^{n+1}+6(-3)^n$$

$$=(-3)^n\left[-27+21+6\right]=0$$

Similarly check that 2^n is a solution.

Casoration $W(n)$

$$= \begin{vmatrix} 1 & (-3)^n & 2^n \\ 1 & (-3)^{n+1} & 2^{n+1} \\ 1 & (-3)^{n+2} & 2^{n+2} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & (-3)^n & 2^n \\ 0 & (-3)^{n+1} - (-3)^n & 2^{n+1} - 2^n \\ 0 & (-3)^{n+2} - (-3)^n & 2^{n+2} - 2^n \end{vmatrix}$$

$$= \begin{vmatrix} (-3)^n(-4) & 2^n \\ (-3)^n 8 & 2^n \cdot 3 \end{vmatrix}$$

$$= (-3)^n 2^n (-12-8) = -20(-6)^n$$

$\Rightarrow W(0) = -20 \neq 0$

$\Rightarrow$ The solution form a fundamental set.

(can choose n_0 s.t. $W(n_0)$ is easier)

H. W.

(1) Verify that $\{n, 2^n\}$ is a fundamental set of solutions of the equation

$$x(n+2) - \frac{3n-2}{n-1}\, x(n+1) + \frac{2n}{n-1}\, x(n) = 0$$

H.W.

(2) Check whether the functions

$\{5^n, \; n5^n, \; n^2 5^n\}$,

$\{0, 3^n, 7^n\}$ are

linearly dependent or not.

(3) Show that,

$1, n, n^2$ are the solutions

of $x(n+3) - 3x(n+2) + 3x(n+1) - x(n) = 0$

Hence find the general solution.

Some Special Topics

Stirling's approximation to n!

We wish to calculate the factorials
After some numbers, it becomes
too much complicated (eg 60!, 80! etc)
The only way is to multiply all values
upto the given numbers

To overcome that difficulty, we can use Stirling's formula to calculate approximate value of n!

$$\boxed{n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

* Derive the Stirling's approximate formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Ans: We will use Gamma function.

$$\underset{n!}{\Gamma(n+1)} = \int_0^\infty e^{-x} x^n \, dx$$

put $y = \frac{x}{n}$ to get

$$\Gamma(n+1) = n! = \int_0^\infty e^{-ny} (ny)^n \, n \, dy$$

$$\Rightarrow n! = n^{n+1}\int_0^\infty e^{-ny}\, y^n \, dy$$

$$= n^{n+1}\int_0^\infty e^{-ny}\, e^{n\log y}\, dy$$

$$= n^{n+1}\int_0^\infty e^{n(\log y - y)}\, dy$$

Now we shall use Laplace's method:

$$\int_a^b e^{M f(x)}\, dx \approx \sqrt{\frac{2\pi}{M\,|f''(x_0)|}}\; e^{M f(x_0)}$$

where M is a large number, the limits a, b may be infinite and $f(x)$ has unique global maximum at x_0

$$\Rightarrow n! = n^{n+1} \int_0^\infty \exp\left(n(\log y - y)\right) dy$$

$$\simeq n^{n+1} \cdot \sqrt{\frac{2\pi}{n\,|f''(1)|}}\; e^{n f(1)}$$

where $f(y) = \log y - y$

$$\simeq n^{n+1} \cdot \sqrt{\frac{2\pi}{n \cdot |-1|}}\; e^{-n}$$

$$\Rightarrow n! \simeq n^{n+1}\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

[This is Stirling's approximation to $n!$]

Rest of the classes,
We will review the
previous topics,
solve problems
& cover some previous
topics in details,
if time permits.